ELSEVIER
Theoretical Computer Science 170 (1996) 145–171
Theoretical Computer Science

Metric completion versus ideal completion

Mila E. Majster-Cederbaum *, Christel Baier

Fakultät für Mathematik und Informatik, Universität Mannheim, Lehrstuhl Prakt., Informatik II, W-6800 Mannheim, Germany

Received May 1995; revised December 1995
Communicated by M. Nivat

Abstract

Complete partial orders have been used for a long time for defining semantics of programming languages. In the context of concurrency de Bakker and Zucker (1982) proposed a metric setting for handling concurrency, recursion and nontermination, which has proved to be very successful in many applications. Starting with a semantic domain D for 'finite behaviour' we investigate the relation between the ideal completion $Idl(D)$ and the metric completion which are both suitable to model recursion and infinite behaviour. We also consider the properties of semantic operators.

1. Introduction

In order to provide denotational semantics to programming languages complete partial orders have been successfully used to model recursive or infinite behaviour of programs. In the context of concurrency de Bakker and Zucker [5] (going back to ideas of M. Nivat) proposed to use complete metric spaces in order to model the behaviour of recursive or infinite concurrent systems. Some semantic domains for modelling concurrent systems, e.g. event structures, trees, pomsets and strings, can be endowed with both a metric and a partial order structure. One way of looking at defining semantics is that one first provides a semantic domain for 'finite behaviour' and secondly uses a completion technique to obtain a domain for 'infinite behaviour'. In this paper we investigate the connection between metric completion and ideal completion techniques. These results are related to our previous investigations [1–3, 12] and shed light on the question of the influence of the choice of mathematical discipline on semantics. We also discuss similar work which has been done in [6]. Other attempts to 'reconcile' the metric and order approach can be found e.g. in [13, 16].

We assume that D is a semantic domain for nonrecursive programs of a *CCS*-like language as finite strings or (labelled) trees of finite height. $\sqsubseteq$ is a partial order on

* Corresponding author. Tel.: +49 (621) 2923; fax: +49 (621) 2925; e-mail: majster-cederbaum@pi2.informatik.uni-mannheim.de.

PII S0304-3975(95)00262-6

D such that D has a bottom element $\perp$ which either can be the meaning of the *nil* program (the program which does not perform any action) or which represents a totally undefined process. If we have semantic operators on D which are monotone w.r.t. $\sqsubseteq$ then the ideal completion $Idl(D)$ can be used as semantic domain for a denotational cpo semantics which extends the semantics on D for recursive programs. On the other hand if D is endowed with a metric such that the semantic operators are non-distance-increasing resp. contracting we get a denotational metric semantics on the metric completion $\overline{D}$. The question arises in which way the metric and ideal completion are related and how the denotational semantics on $Idl(D)$ resp. $\overline{D}$ are connected. In this paper we answer these questions under the assumption that $(D, \sqsubseteq)$ can be endowed with a finite length. This length induces a metric on D. By a finite length we mean a function which assigns the maximal number of atomic steps to each element x of D which are needed for the execution of x. Here the elements of D are considered as processes. E.g. the length of a finite string is its usual length, the length of a tree is its height. The distance $d(x, y)$ induced by a length counts the maximal number n of steps on which the executions of x and y coincide (and then $d(x, y) = 1/2^n$).

The paper is organized as follows: Section 2 presents the concept of a length and a weight on a pointed poset. The relationship between the metric and ideal completion of a pointed poset with a finite length is shown in Section 3. We show that the metric d on D can be lifted to a metric on $Idl(D)$. In Section 3.1 we present conditions for the completeness of $Idl(D)$ as a metric space and we show that then there exists an isometric embedding

$$\mathscr{I}: \overline{D} \to Idl(D).$$

Section 3.2 deals with the question when $\mathscr{I}$ is an isometry, i.e. when the metric completion can be identified with the ideal completion. Section 3.3 shows that under certain conditions the topology induced by the metric induced by a length coincides with the Lawson topology on $Idl(D)$. In Section 4 we discuss the connection between the canonical extensions of monotone and non-distance-increasing operators on the ideal and the metric completions (Section 4.1) and we get a consistency result for denotational semantics Me_{cms} on $\overline{D}$ and Me_{cpo} on $Idl(D)$ of the form

$$\mathscr{I} \circ Me_{\mathrm{cms}} = Me_{\mathrm{cpo}}$$

(Section 4.2). Section 5 discusses the connection of our approach to related work.

2. Pointed posets with a length or weight

Definition 2.1. Let $(D, \sqsubseteq)$ be a pointed poset (i.e. a partially order set with a bottom element which we denote by $\perp_D$ or shortly $\perp$.)
A *length* on $(D, \sqsubseteq)$ is a function $\rho: D \to \mathbb{N}_0 \cup \{\infty\}$ such that for all $x, y \in D$:
(i) $\rho(x) = 0 \ \Leftrightarrow\ x = \perp_D$
(ii) $x \sqsubseteq y \ \Rightarrow\ \rho(x) \leqslant \rho(y)$

Let $Fin(D,\rho)$ denote the collection of all $y \in D$ such that $\rho(y) < \infty$. For all $x \in D$ we define:

$$\downarrow^n(x) = \{y \in D : y \sqsubseteq x,\ \rho(y) \leqslant n\}, \qquad \downarrow^{\text{fin}}(x) = \bigcup_{n \geqslant 0} \downarrow^n_\rho(x).$$

ρ is called *finite* iff $Fin(D,\rho) = D$, i.e. $\rho(x) < \infty$ for all $x \in D$.
An element $x \in D$ is called *approximable* (w.r.t. ρ) iff x is the least upper bound of $\downarrow^{\text{fin}}(x)$. $\mathcal{M}(D, \sqsubseteq, \rho)$ denotes the set of approximable elements.

Lemma 2.2. *Let $(D, \sqsubseteq)$ be a pointed poset and ρ a length on $(D, \sqsubseteq)$. Then*

$$d[\rho](x,y) = \inf \left\{ \frac{1}{2^n} : \downarrow^n(x) = \downarrow^n(y) \right\}$$

is a pseudo ultrametric on D and an ultrametric on $\mathcal{M}(D, \sqsubseteq, \rho)$. $Fin(D,\rho)$ is a subspace of $\mathcal{M}(D, \sqsubseteq, \rho)$.

In particular, if ρ is a finite length on a pointed poset $(D, \sqsubseteq)$ then $d[\rho]$ is an ultrametric on D. In general, the induced metric space $\mathcal{M}(D, \sqsubseteq, \rho)$ is not complete. In order to ensure the completeness of $\mathcal{M}(D, \sqsubseteq, \rho)$ we need additional assumptions.

Definition 2.3. Let $(D, \sqsubseteq)$ be a pointed poset. A *weight* on $(D, \sqsubseteq)$ is a length ρ on $(D, \sqsubseteq)$ such that for all $x \in D$ and $n \geqslant 0$ the set $\downarrow^n(x)$ has a greatest element which we denote by $x[n]$. $x[n]$ is called the *n-cut* of x w.r.t. ρ.

Let $(D, \sqsubseteq)$ be a cpo, i.e. a pointed poset where each directed subset has a least upper bound. A *continuous weight* on $(D, \sqsubseteq)$ is a weight ρ on $(D, \sqsubseteq)$ such that for each $n \geqslant 0$ the function

$$D \to D,\ x \mapsto x[n]$$

is continuous.

Remark 2.4. Let $(D, \sqsubseteq)$ be a pointed poset and ρ a weight on $(D, \sqsubseteq)$. Then for all $n \geqslant 0$ and $x, y \in D$:

(a) If $x \sqsubseteq y$ then $x[n] \sqsubseteq y[n]$.

(b) $\downarrow^m(x[n]) = \downarrow^m(x)$ for all $0 \leqslant m \leqslant n$ and $d[\rho](x, x[n]) \leqslant 1/2^n$. If $m \geqslant n$ then $\downarrow^m(x[n]) = \{y \in D : y \sqsubseteq x[n]\}$.

(c) $d[\rho](x,y) \leqslant 1/2^n$ if and only if $x[n] = y[n]$.

(d) $x[0] = \bot \sqsubseteq x[1] \sqsubseteq x[2] \sqsubseteq \ldots \sqsubseteq x$

(e) $(x[n])[m] = (x[m])[n] = x[n]$ if $m \geqslant n \geqslant 0$.

Lemma 2.5. *Let ρ be a continuous weight on a cpo $(D, \sqsubseteq)$. Then the induced ultrametric space $(\mathcal{M}(D, \sqsubseteq, \rho)$ is complete and*

$$\lim_{n \to \infty} x_n = \bigsqcup_{n \geqslant 0} x_n$$

for each monotone Cauchy sequence in $\mathcal{M}(D, \sqsubseteq, \rho)$.

The concept of a finite weight can be realized on various domains, e.g.

- finite strings over some alphabet A (endowed with the prefixing ordering and the weight $\rho(x) = |x|$ where $|x|$ means the usual length of the string x)
- trees of finite height endowed with Winskels partial order [18] and the height as underlying weight
- prime event structures of finite depth with Winskels partial order [17] and the depth as underlying weight.

It is interesting that the metrics that are obtained by using these length functions are exactly those metrics that have been proposed independently by various authors, see e.g. [8]. In Section 3.4 we show that Mazurkiewicz traces [14] yield an example for a length which is not a weight.

3. The ideal completion as a metric space

Given a finite length on a pointed poset D the set $\mathscr{P}_\downarrow(D)$ of downward-closed subsets can be endowed with a continuous weight. Hence $\mathscr{P}_\downarrow(D)$ turns into a complete metric space. Then the ideal completion $Idl(D)$ as a subspace of $\mathscr{P}_\downarrow(D)$ is also a metric space. We present conditions which ensure the completeness of $Idl(D)$ and we show that if $Idl(D)$ is complete then the metric completion of D can be embedded into the ideal completion. We also present conditions for the isometry of the metric and ideal completion.

Notation 3.1. Let $(D, \sqsubseteq)$ be a pointed poset and $X \sqsubseteq D$. Then

$$X\downarrow = \{y \in D : y \sqsubseteq x \text{ for some } x \in X\}$$

if $X \neq \emptyset$ and $\emptyset \downarrow = \{\bot\}$. X is called *downward-closed* iff $X \downarrow = X$. Let

$$\mathscr{P}_\downarrow(D) = \{X \subseteq D : X \downarrow = X\}$$

denote the set of downward-closed subsets of D. If $x \in D$ then we put

$$x\downarrow = \{x\}\downarrow .$$

X is called *directed* iff for all $x, y \in X$ there exists an upper bound of x and y in X, i.e. there exists $z \in X$ with $x \sqsubseteq z$ and $y \sqsubseteq z$. X is called an *ideal* iff X is downward-closed and directed.

$$Idl(D) = \{X \subseteq D : X \text{ is an ideal}\}$$

denotes the set of ideals of $(D, \sqsubseteq)$. $(\mathscr{P}_\downarrow(D), \subseteq)$ and $(Idl(D), \subseteq)$ are cpo's. The later is called the *ideal completion* of $(D, \sqsubseteq)$. If A is a directed set in $\mathscr{P}_\downarrow(D)$ or $Idl(D)$ then $\bigcup_{X \in A} X$ is the least upper bound of A.

Definition 3.2. If ρ is a length on a pointed poset $(D,\sqsubseteq)$ then $\rho_{\downarrow}: \mathscr{P}_{\downarrow}(D) \to \mathbb{N}_0 \cup \{\infty\}$ is given by:

$$\rho_{\downarrow}(X) = \sup\{\rho(x): x \in X\}$$

If ρ is a length on a pointed poset $(D,\sqsubseteq)$ then $(\mathscr{P}_{\downarrow}(D),\subseteq)$ is a cpo and $\rho_{\downarrow}$ a continuous weight on $(\mathscr{P}_{\perp}(D),\subseteq)$. The n-cut of $X \in \mathscr{P}_{\downarrow}(D)$ is

$$X[n] = \{x \in X: \rho(x) \leqslant n\}.$$

If ρ is finite then all elements of $\mathscr{P}_{\downarrow}(D)$ are approximable. By Lemma 2.5 we obtain:

Lemma 3.3. *Let ρ be a finite length on a pointed poset $(D,\sqsubseteq)$. Then $(\mathscr{P}_{\downarrow}(D), d[\rho_{\downarrow}])$ is a complete metric space. The metric $d[\rho_{\downarrow}]$ on $\mathscr{P}_{\downarrow}(D)$ is given by the formula*:

$$d[\rho_{\downarrow}](X,Y) = \inf\left\{\frac{1}{2^n}: X[n] = Y[n]\right\}$$

Remark 3.4. Let ρ be a finite length on a pointed poset $(D,\sqsubseteq)$. If (X_n) is a Cauchy sequence in $(\mathscr{P}_{\downarrow}(D), d[\rho_{\downarrow}])$ then there exists a sequence $(n_k)_{k\geqslant 0}$ of natural numbers with $n_0 < n_1 < n_2 < \cdots$ and

$$d[\rho_{\downarrow}](X_n,X_m) \leqslant \frac{1}{2^k}$$

for all $n,m \geqslant n_k$. Then:

$$\lim_{n\to\infty} X_n = \bigcup_{k\geqslant 0} X_{n_k}[k]$$

Proof. Let $X = \cup X_{n_k}[k]$. It is clear that X is downward-closed. Since $d[\rho_{\downarrow}](X_n, X_{n_m}) \leqslant 1/2^m$ for all $n \geqslant n_m$ we have: $X_n[m] = X_{n_m}[m]$ for all $n \geqslant n_m$. In particular:

$$X_{n_k}[k][m] = X_{n_k}[m] = X_{n_m}[m]$$

for all $k \geqslant m \geqslant 0$. If $0 \leqslant k < m$ then $n_k < n_m$. Hence

$$X_{n_k}[k][m] = X_{n_k}[k] = X_{n_m}[k] \subseteq X_{n_m}[m].$$

Therefore:

$$X[m] = \left(\bigcup_{k\geqslant 0} X_{n_k}[k]\right)[m] = \bigcup_{k\geqslant 0} X_{n_k}[k][m] = X_{n_m}[m] = X_n[m]$$

for all $n \geqslant n_m$ and $m \geqslant 0$. We conclude that $d[\rho_{\downarrow}](X,X_n) \leqslant 1/2^m$ for all $n \geqslant n_m$. Hence $\lim X_n = X$. $\square$

Remark 3.5. Let ρ be a finite length on a pointed poset $(D,\sqsubseteq)$. If (x_n) is a sequence in D with $d[\rho](x_n,x_m) \leqslant 1/2^n$ for all $m \geqslant n \geqslant 0$ then $(x_n \downarrow)$ is a Cauchy sequence in $(\mathscr{P}_{\downarrow}(D), d[\rho_{\downarrow}])$ and

$$\lim_{n\to\infty} x_n \downarrow = \bigcup_{n\geqslant 0} \downarrow^n(x_n).$$

Proof. Follows by Remark 3.4 and the fact that $(x_n\downarrow)[n] = \downarrow^n (x_n)$. $\square$

Notation 3.6. Let ρ be a length on $(D, \sqsubseteq)$. Then ρ^* denotes the restriction of $\rho_\downarrow$ on $Idl(D)$. d_ρ^* denotes the restriction of $d[\rho_\downarrow]$ to the ideal completion $Idl(D)$ of D.

If ρ is a finite length on $(D, \sqsubseteq)$ then ρ^* is a length on $(Idl(D), \subseteq)$, but in general not a weight. All elements of $Idl(D)$ are approximable and we have:

$$d_\rho^* = d[\rho^*]$$

$(Idl(D), d_\rho^*)$ is a metric space which is in general incomplete. Since $(Idl(D), d_\rho^*)$ is a subspace of $(\mathscr{P}_\downarrow(D), d[\rho_\downarrow])$ we get by Remark 3.4: If (X_n) is a Cauchy sequence in $(Idl(D), \subseteq)$ with $d_\rho^*(X_n, X_m) \leqslant 1/2^n$ for all $m \geqslant n \geqslant 0$ then $\lim X_n$ exists in $Idl(D)$ if and only if the set

$$X = \bigcup_{n \geqslant 0} X_n[n]$$

is directed. In this case, X is an ideal and the limit of (X_n) in $Idl(D)$.

Notation 3.7. Let (M, d) be a metric space. Then the metric completion of (M, d) is denoted by $(\overline{M}, \overline{d})$. We assume that $M \subseteq \overline{M}$ and that d is the restriction of $\overline{d}$ on M.

If (M, d) and (N, d') are metric spaces and $f: M \to N$ a non-distance-increasing function then $\overline{f}$ denotes the unique non-distance-increasing function $\overline{M} \to \overline{N}$ with $\overline{f}(x) = f(x)$ for all $x \in M$.

Lemma 3.8. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$, $d = d[\rho]$. Then*

$$\imath : D \to Idl(D), \qquad \imath(x) = x\downarrow$$

is an isometric embedding of the metric space $(D, d[\rho])$ into the metric space $(Idl(D), d_\rho^*)$ and hence the canonical extension

$$\overline{\imath} : \overline{D} \to \overline{Idl(D)}$$

of $\imath$ is an isometric embedding.

3.1. The completeness of the ideal completion as a metric space

One might ask, if $\overline{\imath}(\overline{D})$ is always contained in $Idl(D)$. The following example shows that, in general, $(Idl(D), d_\rho^*)$ is not complete and $\overline{\imath}(\overline{D})$ is not contained in $Idl(D)$. Consider the pointed poset $(D, \sqsubseteq)$ where

$$D = \{\bot\} \cup \{x_n, y_n, z_n : n \geqslant 1\}$$

and where $\sqsubseteq$ is the smallest partial order on D which satisfies:

$$\perp \sqsubseteq z_1 \sqsubseteq z_2 \sqsubseteq \ldots, \qquad \perp \sqsubseteq y_1 \sqsubseteq y_2 \sqsubseteq \ldots$$

$$z_n, y_n \sqsubseteq x_n \quad \forall n \geqslant 1$$

Let $\rho : D \to \mathbb{N}_0$ be given by:

$$\rho(\perp) = 0, \qquad \rho(z_n) = \rho(y_n) = n, \qquad \rho(x_n) = n + 1$$

Then ρ is a finite length on $(D, \sqsubseteq)$. Since

$$\downarrow^n (x_m) = \{\perp\} \cup \{y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n\}$$

for all $m \geqslant n \geqslant 1$ we have:

$$d_\rho^*(x_n\downarrow,\ x_m\downarrow) = d[\rho](x_n, x_m) \leqslant \frac{1}{2^n} \quad \forall m \geqslant n \geqslant 1$$

Hence $(x_n\downarrow)$ is a Cauchy sequence in $(Idl(D), d_\rho^*)$. Now we assume that $(Idl(D), d_\rho^*)$ is complete. Then $X = \lim x_n\downarrow$ exists in $Idl(D)$. By Remark 3.5:

$$X = \bigcup_{n \geqslant 0} \downarrow^n (x_n) = \{\perp\} \cup \{y_n, z_n : n \geqslant 1\}$$

But the elements $y_1, z_1 \in X$ do not have an upper bound in X. Contradiction! Hence $(Idl(D), d_\rho^*)$ is incomplete. $\square$

Theorem 3.9. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$, $d = d[\rho]$, such that for all $x, y \in D$: If $\{x, y\}$ is bounded in D (i.e. there exists $z \in D$ with $x \sqsubseteq z$ and $y \sqsubseteq z$) then the least upper bound $x \sqcup y$ of x and y exists in D. Then*:

$(Idl(D), d_\rho^*)$ is a complete metric space (and $\bar{\imath} : \overline{D} \to Idl(D)$ is an isometric embedding).

Proof. Let (X_n) be a Cauchy sequence of ideals. W.l.o.g. $d_\rho^*(X_n, X_m) \leqslant 1/2^n$ for all $m \geqslant n \geqslant 0$. (Otherwise we deal with a subsequence (X_{n_k}) of (X_n).) Then

$$X_n[n] = X_m[n] \subseteq X_m[m]$$

for all $m \geqslant n \geqslant 0$. By Remark 3.4:

$$X = \bigcup_{n \geqslant 0} X_n[n]$$

is the limit of (X_n) in $(\mathscr{P}_\downarrow(D), d[\rho_\downarrow])$. We have to show that X is an ideal. (Then $X = \lim X_n$ in $(Idl(D), d_\rho^*)$.)

X is downward-closed (since the sets $X_n[n]$ are downward-closed). Now we show that X is directed: Let $x, y \in X$. Then there exists $n \geqslant 0$ with $x, y \in X_n[n]$. Since X_n is directed there exists $w \in X_n$ with $x, y \sqsubseteq w$. By assumption $z = x \sqcup y$ exists.

If $m \geqslant n$ then $x, y \in X_n[n] \subseteq X_m[m]$. Since X_m is directed there exists $w_m \in X_m$ with $x, y \sqsubseteq w_m$. Then:

$$z \sqsubseteq w_m \quad \forall m \geqslant n$$

Since X_m is downward-closed we have $z \in X_m$ for all $m \geqslant n$. Let

$$m = \max\{n, \rho(z)\}.$$

Then $z \in X_m[m] \subseteq X$. $\square$

In Section 3.4 we show that Theorem 3.9 can be used to obtain the metric completeness of the ideal completion of (finite) Mazurkiewicz traces.

Notation 3.10. Let $(D, \sqsubseteq)$ be a pointed poset, $x \in D$ and $U \subseteq D$. U is called *bounded* iff there exists $u_0 \in D$ with $u \sqsubseteq u_0$ for all $u \in U$. In this case u_0 is called an *upper bound* of U. u_0 is called the *least upper bound* of U (and denoted by $\sqcup U$) iff u_0 is an upper bound of U and $u_0 \sqsubseteq v$ for each upper bound v of U.

$(D, \sqsubseteq)$ is called (*finite*) *bounded complete* iff for each (finite) bounded subset U of D the least upper bound $\sqcup U$ of U exists.

Now we assume that $(D, \sqsubseteq)$ is a cpo. An element $x \in D$ is called *compact* iff whenever $x \sqsubseteq \sqcup U$ where U is a nonempty directed subset of D then $x \sqsubseteq u$ for some $u \in U$. $\mathscr{K}(D)$ denotes the set of compact elements of D. $(D, \sqsubseteq)$ is called an *algebraic* cpo iff $(D, \sqsubseteq)$ is a cpo and for each $x \in D$ the set

$$\mathscr{K}(x) = \{y \in \mathscr{K}(D) : y \sqsubseteq x\}$$

is directed with $x = \sqcup \mathscr{K}(x)$. $(D, \sqsubseteq)$ is called *k-bounded complete* iff for each bounded subset K of $\mathscr{K}(D)$ the least upper bound $\sqcup K$ of K exists.

The finite bounded condition of Theorem 3.9 can be rephrased in the light of the following remark.

Remark 3.11. Let $(D, \sqsubseteq)$ be a pointed poset. Then the following are equivalent:
(I) $(D, \sqsubseteq)$ is finite bounded complete.
(II) $(Idl(D) \subseteq)$ is k-bounded complete.
(III) $(Idl(D), \subseteq)$ is bounded complete.

Proof. (III) $\Rightarrow$ (I) is a standard argumentation. In the following we use the well-known fact that an ideal I is a compact element of $Idl(D)$ if and only if $I = x\downarrow$ for some $x \in D$.

(I) $\Rightarrow$ (II): Let K be a subset of $\mathscr{K}(Idl(D))$ and I_0 an upper bound of K. Let

$$A = \{x \in D : x\downarrow \in K\}.$$

Then $K = \{x\downarrow : x \in A\}$ and $A \subseteq I_0$. Since I_0 is directed for each finite subset V of I_0 there exists an upper bound of V in I_0. In particular, each finite subset V of A has an upper bound in I_0. Hence (by condition (II)): For each finite subset V of A the least upper bound $\sqcup V$ exists in D. Let

$$B = \{\sqcup V : V \subseteq A,\ V \text{ finite}\}, \quad J = B\downarrow.$$

Now we show that J is the least upper bound of K in *Idl*(D).

Claim 1. *J is an ideal.*

Proof. It is sufficient to show that B is directed. Let $z, z' \in B$, $z = \sqcup V$, $z' = \sqcup V'$, where V, V' are finite subsets of A. Then

$$W = V \cup V'$$

is a finite subset of A. Let $w = \sqcup W$. Then $w \in B$ and $z \sqsubseteq w$ (since $V \subseteq W$) and $z' \sqsubseteq w$ (since $V' \subseteq W$). I.e. w is an upper bound of z and z' in B.

Claim 2. *J is an upper bound of K in Idl(D).*

Proof. Let $I \in K$. Then $I = x\downarrow$ for some $x \in A$. Since $\{x\}$ is a finite subset of A and $x = \sqcup\{x\}$ we get $x \in B \subseteq J$. Hence

$$I = x\downarrow \subseteq J.$$

Claim 3. *J is the least upper bound of K in Idl(D).*

Proof. Let I be an upper bound of K in *Idl*(D). Then for all $x \in A : x\downarrow \in K$ and therefore $x\downarrow \subseteq I$, i.e. $x \in I$. Hence $A \subseteq I$.

Let $x \in J$. Then $x \sqsubseteq y$ for some $y \in B$. Let V be a finite subset of A with $y = \sqcup V$. Then V is a finite subset of I. Since I is directed there exists an upper bound z of V in I. Then

$$x \sqsubseteq y \sqsubseteq z \in I.$$

Since I is downward-closed we get: $x \in I$. We conclude that $J \subseteq I$.

(II) $\Rightarrow$ (III): We use the fact that *Idl*(D) is an algebraic cpo and show more generally: If D is an algebraic cpo then k-bounded completeness implies bounded completeness.

Let U be a bounded subset of *Idl*(D). Then

$$A = \bigcup_{x \in U} \mathscr{K}(x)$$

is a bounded subset of $\mathscr{K}(D)$. (Note that each upper bound of U is also an upper bound of A.) Since *Idl*(D) is k-bounded complete: $\sqcup A = a$ exists. Then for all $x \in U$:

$$x = \sqcup \mathscr{K}(x) \sqsubseteq a$$

Hence a is an upper bound of U. If z is also an upper bound of U then z is an upper bound of A. Therefore,

$$a = \sqcup A \sqsubseteq z$$

We conclude: $a = \sqcup U$. $\square$

Lemma 3.12. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that for each ideal I and each natural number n the set $I[n]$ is directed. Then*

(a) ρ^* is a continuous weight on $(Idl(D), \subseteq)$.

(b) $(Idl(D), d_\rho^*)$ is a complete metric space (and $\bar{\imath} : \overline{D} \to Idl(D)$ is an isometric embedding).

Proof. It is easy to see that $I[n]$ is the n-cut of I in $(Idl(D), \subseteq)$ w.r.t. ρ^*. I.e. ρ^* is a weight. The function

$$Idl(D) \to Idl(D), \qquad I \mapsto I[n]$$

is continuous. Here we use the fact that if $I = \cup I_m$ then

$$I[n] = \bigcup_{m \geqslant 0} I_m[n].$$

Hence ρ^* is a continuous weight. Since ρ is finite we have $I = \cup I[n]$ for all ideals I. Hence all elements of $Idl(D)$ are approximable. Therefore (b) follows by (a) and Lemma 2.5. $\square$

Theorem 3.13. *Let ρ be a finite weight on a pointed poset $(D, \sqsubseteq)$. Then $(Idl(D), d_\rho^*)$ is a complete metric space and*

$$\bar{\imath} : \overline{D} \to Idl(D)$$

is an isometric embedding.

Proof. By Lemma 3.12 it is sufficient to show that for each ideal I the sets $I[n]$ are directed.

Let I be an ideal, $n \geqslant 0$, $x, y \in I[n]$. Let $w \in I$ be an upper bound of x and y. Since I is downward-closed $w[n] \in I$. Since $\rho(x), \rho(y) \leqslant n$ we get:

$$x = x[n] \sqsubseteq w[n], \qquad y = y[n] \sqsubseteq w[n].$$

I.e. $w[n]$ is an upper bound of x and y in $I[n]$. $\square$

By Theorem 3.13 we get e.g. that the metric completion of trees of finite height is a subspace of the ideal completion (which is Winskels cpo of trees).

3.2. Metric and ideal completion

If $Idl(D)$ is a complete metric space then $\overline{D}$ is a closed subspace of $Idl(D)$ as a metric space but not equal to $Idl(D)$ (see Section 3.4). Hence we will investigate conditions that guarantee equality. We show that under the assumptions of Theorems 3.9 and 3.13, and the additional assumption that the sets $I[n]$ are finite the metric completion of D and the ideal completion of D are isometric.

Lemma 3.14. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that:*
(i) *$(Idl(D), d_\rho^*)$ is a complete metric space.*
(ii) *For all $I \in Idl(D)$ the set $I[n]$ is finite.*
Then $\bar{\imath} : \overline{D} \to Idl(D)$ is an isometry.

Proof. We have to show that $\bar{\imath}$ is surjective.

Let I be an ideal. By assumption (ii) for all $n \geqslant 0$ the n-cut $I[n]$ of I is finite. Since I is directed there exists an upper bound $x_n \in I$ of $I[n]$, i.e. $y \sqsubseteq x_n$ for all $y \in I[n]$. Then

$$\downarrow^n (x_n) = (x_n \downarrow)[n] = I[n].$$

We define by induction on k a subsequence (x_{n_k}) of (x_n) as follows:

In the basis of induction ($k = 0$) we define $n_0 = 0$. In the induction step $k \Rightarrow k+1$ we define

$$n_{k+1} = 1 + \max\{\rho(x_{n_k}), n_k\}.$$

Claim 1. *(x_{n_k}) is a Cauchy sequence in D.*

Proof. For fixed $n \geqslant 0$ we show that there exists $k_m \geqslant 0$ with $d[\rho](x_{n_k}, x_{n_l}) \leqslant 1/2^m$ for all $k,\ l \geqslant k_m$.

Since $x_{n_k} \in I[n_{k+1}]$ we have $x_{n_k} \sqsubseteq x_{n_{k+1}}$. Since $x_{n_k} \in I$ we get:

$$\downarrow^m (x_{n_k}) \subseteq I[m].$$

Hence $y \sqsubseteq x_m$ for all $y \in \downarrow^m (x_{n_k})$ and $k, m \geqslant 0$. I.e. $\downarrow^m (x_{n_k}) \subseteq x_m \downarrow$ for all $k \geqslant 0$. Therefore

$$\downarrow^m (x_{n_0}) \subseteq \downarrow^m (x_{n_1}) \subseteq \cdots \subseteq x_m \downarrow .$$

Let $k = \rho(x_m)$. Then $x_m \downarrow = (x_m \downarrow)[k]$ is a finite set (assumption (ii)). Hence there exists $k_m \geqslant 0$ with

$$\downarrow^m (x_{n_k}) = \downarrow^m (x_{n_{k_m}})$$

for all $k \geqslant k_m$. Therefore $d[\rho](x_{n_k}, x_{n_l}) \leqslant 1/2^m$ for all $k, l \geqslant k_m$.

Claim 2. *$\bar{\imath}(x) = I$ where $x = \lim x_{n_k}$.*

Proof. Let $k_0 < k_1 < \cdots$ be a sequence of natural numbers with

$$d[\rho](x_{n_k}, x_{n_l}) \leqslant \frac{1}{2^m} \quad \forall k, l \geqslant k_m$$

and let $y_m = x_{n_{k_m}}$. Then (y_m) is a subsequence of (x_{n_k}) with $d[\rho](y_n, y_m) \leqslant 1/2^n$ for all $m \geqslant n \geqslant 0$. Hence

$$\lim_{m\to\infty} y_m = \lim_{k\to\infty} x_{n_k} = x$$

and by Remark 3.5

$$\bar{\imath}(x) = \lim_{m\to\infty} y_m\downarrow = \bigcup_{m\geqslant 0} \downarrow^m(y_m).$$

Now we show that $I = \bar{\imath}(x)$.

- If $x \in I$ then $x \in I[m]$ for some $m \geqslant 0$. Since $n_{k_m} \geqslant k_m \geqslant m$:

  $$x \sqsubseteq x_m \sqsubseteq x_{n_{k_m}} = y_m.$$

  Then $x \in \downarrow^m(y_m) \subseteq \bar{\imath}(x)$.
- If $x \in \bar{\imath}(x)$ then $x \in \downarrow^m(y_m)$ for some $m \geqslant 0$. Since $y_m \in I$ and since I is downward-closed we have: $x \in I$. □

Theorem 3.15. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that the following conditions* (i) *and* (ii) *are satisfied*:

(i) *D is finite bounded complete.*

(ii) *For all $I \in Idl(D)$ the set $I[n]$ is finite.*

Then $\bar{\imath} : \overline{D} \to Idl(D)$ is an isometry.

Proof. Follows by Theorem 3.9 and Lemma 3.14. □

In Section 3.4 we show that the condition (ii) of Theorem 3.9 is essential.

Theorem 3.16. *Let ρ be a finite weight on a pointed poset $(D, \sqsubseteq)$ such that for all $I \in Idl(D)$ the n-cut $I[n]$ is a finite set. Then $\bar{\imath} : \overline{D} \to Idl(D)$ is an isometry.*

Proof. Follows by Theorem 3.13 and Lemma 3.14. □

By Theorem 3.16 we get e.g. the well-known result that the metric completion of finite strings over some alphabet A coincides with the ideal completion of finite strings (which is the cpo of infinite strings over A).

3.3. The d_ρ^*-topology and the Lawson topology of Idl(D)

We show that under certain conditions the topology induced by d_ρ^* equals the Lawson topology on $Idl(D)$. We omit the general definitions of the topologies induced by a metric resp. the Lawson topology. They can be found e.g. in [7, 10]. We only specify the open sets w.r.t. the d_ρ^*-topology and the Lawson topology.

- A subset U of $Idl(D)$ is open w.r.t. the d_ρ^*-topology if and only if U can be written as a union of balls

  $$B_n(I) = \{J \in Idl(D) : I[n] = J[n]\}$$

where $n \in \mathbb{N}_0$ and $I \in Idl(D)$.

- Let $(D, \sqsubseteq)$ be a pointed poset such that D is countable. Then a subset U of $Idl(D)$ is open w.r.t. the Lawson topology if and only if

$$U = \bigcup_{i \in A} \left(\bigcap_{j \in B(i)} W_{i,j} \cap \bigcap_{j \in C(i)} V_{i,j} \right)$$

where

1. A is an arbitrary indexing set.
2. For all $i \in A$ the set $B(i)$ is finite and for all $j \in B(i)$ there exists $x \in D$ with

$$W_{i,j} = \{J \in Idl(D) : x \in J\}.$$

3. For all $i \in A$ the set $C(i)$ is finite and for all $j \in C(i)$ there exists $x \in D$ with

$$V_{i,j} = \{J \in Idl(D) : x \notin J\}.$$

If $A = \emptyset$ then $\bigcup_{i \in A} \cdots = \emptyset$ and $\bigcap_{i \in A} \cdots = Idl(D)$.

Theorem 3.17. *Let ρ be a finite length on a countable pointed poset $(D, \sqsubseteq)$ such that for all $n \geqslant 0$ the set*

$$\{x \in D : \rho(x) \leqslant n\}$$

is finite. Then the d_ρ^*-topology coincides with the Lawson topology. I.e. if U is a subset of $Idl(D)$ then U is open w.r.t. the d_ρ^*-topology if and only if U is open w.r.t. the Lawson topology.

Proof. If $x \in D$ then we define:

$$W_x = \{I \in Idl(D) : x \in I\}, \qquad V_x = \{I \in Idl(D) : x \notin I\}.$$

Since the union and finite intersection of open sets is always open it is sufficient to show that:

(i) The sets W_x, V_x are open w.r.t. the d_ρ^*-topology.

(ii) The balls $B_n(I)$ are open w.r.t. the Lawson topology.

For (i): Let $x \in D$, $\rho(x) = n$. If $I \in W_x$ then $x \in I[n]$. Hence for all $J \in B_n(I)$:

$$x \in I[n] = J[n] \subseteq J.$$

I.e. $J \in W_x$. Therefore $B_n(I) \subseteq W_x$. We conclude

$$W_x = \bigcup \{B_n(I) : I \in W_x\}.$$

Hence W_x is open w.r.t. the d_ρ^*-topology. Similarly it can be shown that

$$V_x = \{B_n(I) : I \in V_x\}$$

is open w.r.t. the d_ρ^*-topology.

For (ii): Let $I \in Idl(D)$ and $n \geqslant 0$. Let $U = W \cup V$ where

$$W = \bigcap_{x \in I[n]} W_x, \qquad V = \bigcap_{x \in C} V_x$$

and where $C = \{x \in D \backslash I : \rho(x) \leqslant n\}$.

By assumption the sets $I[n]$ and C are finite. Hence U is open w.r.t. the Lawson topology. Now we show that $B_n(I) = U$.

1. If $J \in B_n(I)$ then $I[n] = J[n]$. Hence:
- $J \in W_x$ for all $x \in I[n]$. Therefore $J \in W$.
- $J \in V_x$ for all $x \in C$. Therefore $J \in V$.

We conclude $J \in W \cap V = U$.

2. If $J \in U$ then $J \in W$ and $J \in V$. Hence:
- Since $J \in W$ we have: If $x \in I[n]$ then $J \in W_x$, i.e. $x \in J$. Hence $I[n] \subseteq J$. We conclude $I[n] \subseteq J[n]$.
- Since $J \in V$ we have: If $x \in C$ then $J \in V_x$, i.e. $x \notin J$. Therefore $J[n] \subseteq I$. We conclude $J[n] \subseteq I[n]$.

We get: $I[n] = J[n]$ and therefore $J \in B_n(I)$. Hence $U \subseteq B_n(I)$. □

By Theorem 3.17 we get that the topology induced by the 'standard metric' on (finite or infinite) strings coincides with the Lawson topology.

Remark 3.18. If the condition $\{x \in D : \rho(x) \leqslant n\}$ finite is violated we get the first part of the proof of Theorem 3.17 that the d_ρ^*-topology is finer than the Lawson topology (i.e. open w.r.t. the Lawson topology implies open w.r.t. the d_ρ^*-topology). But in general the d_ρ^*-topology does not agree with the Lawson topology. E.g. consider the pointed poset $(D, \sqsubseteq)$ where

$$D = \{\perp\} \cup \{x_n : n \geqslant 1\}$$

and

$$\perp \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$$

Let ρ be given by:

$$\rho(x) = \begin{cases} 0 & \text{if } x = \perp, \\ 1 & \text{otherwise.} \end{cases}$$

Then d_ρ^* is the discrete metric on $Idl(D)$. Hence all subsets of $Idl(D)$ are open w.r.t. the d_ρ^*-topology. Since D is totally ordered we have for each nonempty finite subset A of D:

$$\bigcap_{x \in A} W_x = W_{x_m}, \qquad \bigcap_{x \in A} V_x = V_{x_k},$$

where $m = \max\{n : x_n \in A\}$ and $k = \min\{n : x_n \in A\}$. (Here $x_0 = \perp$.) Hence a subset U of $Idl(D)$ is open w.r.t. the Lawson topology if and only if U is empty or U can be written as union of sets of the form $W \cap V$ where $W = Idl(D)$ or $W = W_x$ for some $x \in D$ and $V = Idl(D)$ or $V = V_y$ for some $y \in D$. Now we consider

$$U = \{D\}.$$

U is open w.r.t. the d_ρ^*-topology (since all subsets of $Idl(D)$ are open w.r.t. the d_ρ^*-topology). We show that U is not open w.r.t. the Lawson topology:

If U would be open w.r.t. the Lawson topology then $U = W \cap V$ where $W = Idl(D)$ or $W = W_x$ for some $x \in D$ and $V = Idl(D)$ or $V = V_y$ for some $y \in D$. (Here we use the fact that U consists of a single element.) Since $D \in V$ we conclude $V = Idl(D)$. Therefore $U = W$. The case $W = W_x$ is not possible since then

$$x \downarrow \in W \setminus U.$$

Hence $W = Idl(D)$. Then $\{D\} = U = Idl(D)$. Contradiction. □

3.4. *Example*: *Mazurkiewicz traces*

In order to give an example for a finite length which is not a weight we consider the pointed poset of traces in the sence of [14]. In [11] the concept of (finite) traces is generalized to traces of length up to ω by dealing with the order resp. metric completion. We recall the basic notions of trace theorem as in [11, 14]:

Let (A, ι) be a *concurrent alphabet*, i.e. A is a set of actions and ι an irreflexive and symmetric relation on A (called *independency*). A *trace* is an equivalence class $[x]$ of a finite string x over A where the underlying equivalence relation is the reflexive, transitive closure of $\equiv$ which is given by:

$$x \equiv y \quad :\Leftrightarrow \quad \exists \alpha, \beta \in A, \quad z, w \in A^* : \alpha \iota \beta \wedge x = z\alpha\beta w \wedge y = z\beta\alpha w.$$

If $\sigma = [x]$ is a trace then $|\sigma| = |x|$ where $|x|$ means the usual length of x. In the following Θ^* denotes the set of traces w.r.t. a fixed concurrent alphabet (A, ι) and $\sqsubseteq$ means the lifting of the prefixing ordering on A^* to Θ^*. I.e.

$$[x] \sqsubseteq [y] \quad :\Leftrightarrow \quad \exists x', y', z \in A^* : x' \equiv x \wedge y' \equiv y \wedge y' = x'z$$

In $n \in \mathbb{N}$ then we put:

$$\sigma^{(n)} = \{\sigma' \in \Theta^* : \sigma' \sqsubseteq \sigma, |\sigma'| \leqslant n\}$$

[11] considers the metric

$$d(\sigma, \tau) = \inf \left\{ \frac{1}{2^n} : \sigma^{(n)} = \tau^{(n)} \right\}$$

on Θ^* and the metric

$$d^*(I, J) = \inf \left\{ \frac{1}{2^n} : I^{(n)} = J^{(n)} \right\}$$

on $Idl(\Theta^*)$ where $I^{(n)} = \{\sigma^{(n)} : \sigma \in I\}$. [11] shows:

(I) Every bounded pair in Θ^* has a least upper bound.

(II) If A is countable then $Idl(\Theta^*)$ is bounded complete.
(III) If A is countable then $(Idl(\Theta^*), d^*)$ is complete.
(IV) If A is finite then $(\overline{\Theta^*}, \overline{d})$ and $(Idl(\Theta^*), d^*)$ are isometric.
(V) If A is infinite then $(\overline{\Theta^*}, \overline{d})$ and $(Idl(\Theta^*), d^*)$ are not isometric.
(VI) If A finite then d^*-topology and the Lawson topology on $Idl(\Theta^*)$ are the same.

The results of [11] fit nicely in our framework: We consider the finite length

$$\rho : \Theta^* \to \mathbb{N}_0, \qquad \rho(\sigma) = |\sigma|.$$

Then $\downarrow^n(\sigma) = \sigma^{(n)}$ and therefore $d = d[\rho]$. If I is an ideal then $I^{(n)} = I[n]$. Hence $d^* = d^*_\rho$.

If $\iota \neq \emptyset$ then ρ is not a weight, e.g. if $\alpha, \beta \in A, \alpha\iota\beta$ then

$$\downarrow^1([\alpha\beta]) = \{\bot, [\alpha], [\beta]\}$$

does not contain a greatest element since $[\alpha], [\beta]$ are incomparable.

Please note that result (II) is strengthened by our Remark 3.11: (I) and Remark 3.11 ensure that $Idl(\Theta^*)$ is always bounded complete. Also result (III) is strengthened by our Theorem 3.9. By (I) and Theorem 3.9 we may conclude that $(Idl(\Theta^*), d^*)$ is metrically complete independent of the cardinality of A.

If A is finite then $\{x \in A^* : |x| \leqslant n\}$ and then also $\{\sigma \in \Theta^* : \rho(\sigma) \leqslant n\}$ are finite. In particular, for each ideal I the set $I[n]$ is finite. Hence the conditions of Theorems 3.15 and 3.17 are satisfied. I.e. the results (IV) and (VI) are special cases of Theorem 3.15 resp. Theorem 3.17. The result (V) shows that condition (ii) of Theorem 3.15 is essential.

4. Denotational semantics on the metric and ideal completion

In order to compare the cpo denotational semantics Me_{cpo} on $Idl(D)$ with a metric denotational semantics Me_{cms} on $\overline{D}$ we assume that the semantic operators on $Idl(D)$ resp. $\overline{D}$ are the canonical extensions of semantic operators on D. We discuss the relationship between the canonical extensions and present conditions for a consistency result of the form $\bar{\imath} \circ Me_{\mathrm{cms}} = Me_{\mathrm{cpo}}$.

4.1. The canonical extensions of operators on the metric and ideal completion

Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$. If $f : D \to D$ is a function which is monotone w.r.t. $\sqsubseteq$ and non-distance-increasing w.r.t. $d[\rho]$ then the question arises in which way the canonical extension

$$f^* : Idl(D) \to Idl(D), \qquad f^*(I) = f(I)\downarrow$$

of f on the ideal completion and the canonical extension

$$\overline{f} : \overline{D} \to \overline{D}, \qquad \overline{f}\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n)$$

of f on the metric completion are related. If $(Idl(D), d_\rho^*)$ is complete one might suppose that

$$\bar{\imath} \circ \overline{f} = f^* \circ \bar{\imath}.$$

In the case that ρ is a weight this is true (see Lemma 4.2) but in general this is wrong (see Example 4.1). Nevertheless in the general case we have $\bar{\imath}(fix(\overline{f})) = lfp(f^*)$ when we assume that $\overline{f}$ is contracting (Lemma 4.4). Here $fix(\overline{f})$ means the unique fixed point of $\overline{f}$ (Banach's fixed point theorem) and $lfp(f^*)$ means the least fixed point of f^* (Tarski's fixed point theorem).

Example 4.1. Consider the pointed poset $(D, \sqsubseteq)$ where

$$D = \{\bot, x\} \cup \{x_n, y_n : n \geqslant 1\}$$

and where $\sqsubseteq$ is the smallest partial order on D which satisfies:

$$\bot \sqsubseteq x \quad \text{and} \quad \bot \sqsubseteq y_1 \sqsubseteq y_2 \sqsubseteq \cdots \sqsubseteq x_n \quad \forall n \geqslant 1.$$

The elements x_n, $n \geqslant 1$, are pairwise incomparable. Let $\rho : D \longrightarrow \mathbb{N}_0$ be given by:

$$\rho(\bot) = 0, \qquad \rho(x_n) = n + 1, \qquad \rho(y_n) = \rho(x) = 1.$$

Then ρ is a finite length on $(D, \sqsubseteq)$. It is easy to see: If $u, v \in D$ have an upper bound in D then $u \sqcup v$ exists in D. By Theorem 3.9 $(Idl(D), d_\rho^*)$ is a complete metric space for each finite length ρ on D.

We have for all $m \geqslant n \geqslant 1$ and $k \geqslant 1$:

$$\downarrow^n(x) = \{\bot, x\}, \qquad \downarrow^n(y_k) = \{\bot, y_1, y_2, \ldots, y_k\}, \qquad \downarrow^n(x_m) = \{\bot\} \cup \{y_n : n \geqslant 1\}.$$

Hence $d[\rho](x_n, x_m) = 1/2^n$ for all $m \geqslant n \geqslant 1$ and $d[\rho](y_k, x_m) = d[\rho](x, x_m) = 1$ for all $k, m \geqslant 1$. Therefore the function $f : D \to D$,

$$f(z) = \begin{cases} x & \text{if } z = x_m \text{ for some } m \geqslant 1 \\ \bot & \text{otherwise} \end{cases}$$

is monotone and non-distance-increasing. Since (x_n) is a Cauchy sequence of D the limit of (x_n) exists in $\overline{D}$. Let

$$\xi = \lim_{n \to \infty} x_n \in \overline{D}.$$

Then by Remark 3.5:

$$\imath(\xi) = \lim_{n \to \infty} x_n \downarrow = \bigcup_{n \geqslant 1} \downarrow^n(x_n) = \{\bot\} \cup \{y_n : n \geqslant 1\}.$$

Hence $f^*(\imath(\xi)) = \{\bot\}$. On the other hand:

$$\overline{f}(\xi) = \lim_{n \to \infty} f(x_n) = x.$$

Therefore:

$$\imath(\overline{f}(\xi)) = x\downarrow = \{\bot, x\} \neq \{\bot\} = f^*(\imath(\xi)).$$

Lemma 4.2. *Let ρ be a finite weight on a pointed poset $(D, \sqsubseteq)$. If $f : D \to D$ is monotone and non-distance-increasing then*

$$\bar{\imath} \circ \bar{f} = f^* \circ \bar{\imath}$$

and f^* is non-distance-increasing w.r.t. d_ρ^*. If f is contracting then also f^* is contracting.

Proof. Let $x \in \overline{D}$. Then $x = \lim x_n$ for some sequence (x_n) in D with

$$d[\rho](x_n, x_m) \leqslant \frac{1}{2^n}$$

for all $m \geqslant n \geqslant 0$. Then $x_n[n] = x_m[n]$ for all $m \geqslant n \geqslant 0$ and

$$\bar{f}(x) = \lim_{n\to\infty} f(x_n).$$

Since f is non-distance-increasing we have:

$$d[\rho](f(x_n), f(x_m)) \leqslant d[\rho](x_n, x_m) \leqslant \frac{1}{2^n}$$

for all $m \geqslant n \geqslant 0$. By Remark 3.5:

$$\bar{\imath}(\bar{f}(x)) = \lim_{n\to\infty} f(x_n)\downarrow = \bigcup_{n\geq 0} \downarrow^n(f(x_n)) = \bigcup_{n\geqslant 0} f(x_n)[n]\downarrow.$$

On the other hand (also by Remark 3.5):

$$f^*(\bar{\imath}(x)) = f^*\left(\bigcup_{n\geqslant 0} \downarrow^n(x_n)\right) = \bigcup_{n\geqslant 0} f(\downarrow^n(x_n))\downarrow = \bigcup_{n\geqslant 0} f(x_n[n])\downarrow.$$

Since f is non-distance-increasing and since $d[\rho](x_n, x_n[n]) \leqslant 1/2^n$ we get:

$$d[\rho](f(x_n), f(x_n[n])) \leqslant \frac{1}{2^n}.$$

I.e. $f(x_n)[n] = f(x_n[n])[n]$. Since

$$f(x_n[n])[n]\downarrow \subseteq f(x_n)[n])\downarrow$$

we get:

$$\bar{\imath}(\bar{f}(x)) = \bigcup_{n\geqslant 0} f(x_n)[n]\downarrow = \bigcup_{n\geqslant 0} f(x_n[n])[n]\downarrow \subseteq f^*(\bar{\imath}(x)).$$

On the other hand: If $y \in f^*(\bar{\imath}(x))$ then $y \sqsubseteq f(x_n[n])$ for some $n \geqslant 0$. Let $m = \max\{\rho(y), n\}$. Since f is monotone and $x_n[n] = x_m[n] \sqsubseteq x_m[m]$ we get:

$$y \sqsubseteq f(x_n[n]) = f(x_m[n]) \sqsubseteq f(x_m[m]).$$

Hence $y \sqsubseteq f(x_m[m])[m]$. Therefore $y \in \bar{\imath}(\bar{f}(x))$. We conclude:

$$\bar{\imath}(\bar{f}(x)) = f^*(\imath(x)).$$

Now we show that f^* is non-distance-increasing w.r.t. d_ρ^*. Let $I,J \in Idl(D)$, $d_\rho^*(I,J) = 1/2^n$. Then for all $y \in f^*(I)[n]$:

$$y \sqsubseteq f(x) \quad \text{for some } x \in I.$$

Since $x[n] \in I[n] = J[n]$ and $\rho(y) \leqslant n$ we get:

$$y \sqsubseteq f(x)[n] = f(x[n])[n] \in f^*(J)[n].$$

We conclude: $f^*(I)[n] \subseteq f^*(J)[n]$. By symmetry we get $f^*(I)[n] = f^*(J)[n]$. Therefore,

$$d_\rho^*(f^*(I), f^*(J)) \leqslant \frac{1}{2^n} = d_\rho^*(I,J).$$

Similarly, it can be shown that f^* is contracting if f is contracting. $\square$

Remark 4.3. Example 4.1 also shows that in general f^* is not non-distance-increasing. Let $A, \sqsubseteq, \rho$ and f be as in Example 4.1 and

$$I = \{\bot, y_1, y_2, \ldots\}, \qquad J = x_1 \downarrow = I \cup \{x_1\}.$$

Then $f^*(I) = \{\bot\}$, $f^*(J) = \{\bot, x\}$. Hence

$$d_\rho^*(f^*(I), f^*(J)) = 1 > \tfrac{1}{2} = d_\rho^*(I,J).$$

Lemma 4.4. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$. Let $f : D \to D$ be a function which is monotone and contracting w.r.t. $d[\rho]$. Then $\bar{f}$ is contracting with contracting constant $\frac{1}{2}$ and*

$$\bar{\imath}(fix(\bar{f})) = lfp(f^*).$$

Proof. Let $d = d[\rho]$. For all $x, y \in D$, $x \neq y$, we have: There exists $n \geqslant 0$ with $d(x,y) = 1/2^n$. Since f is contracting:

$$d(f(x), f(y)) < d(x,y) = \frac{1}{2^n}.$$

Hence $d(f(x), f(y)) \leqslant \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2} d(x,y)$. I.e. f and then also $\bar{f}$ are contracting with contracting constant $\frac{1}{2}$.

Let $x_0 = \bot$, $x_{n+1} = f(x_n) = \bar{f}(x_n)$. By Banach's fixed point theorem:

$$d(x_n, x_m) = \frac{1}{2^n} \quad \forall m \geqslant n \geqslant 0$$

and

$$fix(\bar{f}) = \lim_{n \to \infty} x_n.$$

By Remark 3.5:

$$\bar{\imath}(\mathit{fix}(\bar{f})) = \bigcup_{n\geqslant 0} \downarrow^n (x_n).$$

By Tarski's fixed point theorem:

$$\mathit{lfp}(f^*) = \bigcup_{n\geqslant 0} I_n$$

where $I_0 = \{\bot\}$ and $I_{n+1} = f^*(I_n) = f(I_n)\downarrow$. We show by induction on n that

$$I_n = x_n \downarrow.$$

In the basis of induction ($n = 0$) we have: $I_0 = \{\bot\} = \bot\downarrow = x_0\downarrow$.

In the induction step $n \Rightarrow n+1$ we assume that $I_n = x_n\downarrow$. Since f is monotone we get:

$$I_{n+1} = f(I_n)\downarrow = f(x_n)\downarrow = x_{n+1}\downarrow.$$

Since f is monotone we have $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$. Since ρ is finite we get: If $y \in x_n\downarrow$ then $y \in \downarrow^m(x_m)$ where $m = \max\{\rho(y), n\}$. Hence we get:

$$\bigcup_{n\geqslant 0} x_n\downarrow = \bigcup_{n\geqslant 0} \downarrow^n (x_n).$$

Therefore

$$\bar{\imath}(\mathit{fix}(\bar{f})) = \bigcup_{n\geqslant 0} \downarrow^n (x_n) = \bigcup_{n\geqslant 0} x_n\downarrow = \mathit{lfp}(f^*). \qquad \square$$

4.2. Consistency of denotational semantics on the metric and ideal completion

Our aim is to establish a consistency result for denotational semantics on the complete metric space $\overline{D}$ and the cpo $\mathit{Idl}(D)$. We shortly explain what we mean by a denotational metric semantics on $\overline{D}$ and a denotational cpo semantics on $\mathit{Idl}(D)$. For details of the cpo resp. metric approach to give denotational semantics see e.g. [1, 4, 5, 9].

We assume that D is a semantic domain for nonrecursive programs and that ρ is a finite length on D. We consider a language where recursion is modelled by declarations, i.e. a program is a pair $\langle s, \sigma\rangle$ where s is a statement (which is built from operator symbols like prefixing or sequential composition, nondeterministic choice, parallelism, etc. and process variables) and a declaration σ (i.e. σ is a function which assigns a statement $\sigma(x)$ to each process variable x). $\mathscr{L}$ denotes the set of statements.

For each operator symbol ω in $\mathscr{L}$ let ω_D be a semantic operator on D which is monotone w.r.t. $\sqsubseteq$ and non-distance-increasing/contracting w.r.t. $d[\rho]$. For a fixed declaration σ we may define a mapping

$$F : (\mathscr{L} \to D) \to (\mathscr{L} \to D)$$

by structural induction on $s \in \mathscr{L}$:

$$F(f)(a) = a_D \quad \text{for each constant symbol } a \text{ in } \mathscr{L}$$

$$F(f)(x) = f(\sigma(x)) \quad \text{for each process variable } x$$

$$F(f)(w(s_1,\ldots,s_n)) = \omega_D(F(f)(s_1),\ldots,F(f)(s_n))$$
$$\text{for each } n\text{-ary operator symbol } w \text{ in } \mathscr{L}$$

Similarly we get mappings

$$F_{\text{cms}} : (\mathscr{L} \to \overline{D}) \to (\mathscr{L} \to \overline{D}), \qquad F_{\text{cpo}} : (\mathscr{L} \to \mathit{Idl}(D)) \to (\mathscr{L} \to \mathit{Idl}(D))$$

where we use the canonical extensions $\overline{\omega_D}$ resp. ω_D^* as semantic operators. Since F_{cpo} is continuous we have a denotational cpo semantics on $\mathit{Idl}(D)$:

$$Me_{\text{cpo}} : \mathscr{L} \to \mathit{Idl}(D), \qquad Me_{\text{cpo}} = \mathit{lfp}(F_{\text{cpo}}).$$

Under certain conditions (e.g. the guardedness of the statements $\sigma(x)$ in the sense of [15]) the function F_{cms} is contracting and hence has a unique fixed point. We get a metric denotational semantics on $\overline{D}$:

$$Me_{\text{cms}} : \mathscr{L} \to \overline{D}, \qquad Me_{\text{cms}} = \mathit{fix}(F_{\text{cms}}).$$

We establish the following consistency result:

Theorem 4.5. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$. Then for Me_{cms} and Me_{cpo} as sketched above we obtain*:

$$\bar{\imath} \circ Me_{\text{cms}} = Me_{\text{cpo}}.$$

Proof. By structural induction on $s \in \mathscr{L}$ it can be shown that

$$(\imath \circ F(f))(s) = F_{\text{cpo}}(\imath \circ f)(s)$$

for all functions $f : \mathscr{L} \to D$. By Banach's and Tarski's fixed point theorem:

$$Me_{\text{cms}} = \lim_{n\to\infty} f_n, \qquad Me_{\text{cpo}} = \bigcup_{n \geqslant 0} g_n$$

where $f_0 = \lambda t.\bot$, $f_{n+1} = F_{\text{cms}}(f_n)$ and $g_0 = \lambda t.\{\bot\}$, $g_{n+1} = F_{\text{cpo}}(g_n)$. By induction on n it can be shown that $f_n(\mathscr{L}) \subseteq D$ and $f_{n+1} = F(f_n)$. Hence we get (again by induction on n):

$$\imath \circ f_n = g_n.$$

Now we assume that s is a fixed statement. Let $x_n = f_n(s)$. Then:

$$Me_{\text{cms}}(s) = \lim_{n\to\infty} x_n.$$

Because of Remark 3.5:

$$\bar{\imath}(Me_{\mathrm{cms}}(s)) = \lim_{n\to\infty} x_n\downarrow = \bigcup_{n\geqslant 0} \downarrow^n(x_n)$$

Here we make use of the fact that F_{cms} is contracting with contracting constant $\frac{1}{2}$ and therefore $d[\rho](x_n,x_m)\leqslant 1/2^n$ for all $m\geqslant n\geqslant 0$. Since $\imath\circ f_n = g_n$ we have: $g_n(s) = \imath(f_n(s)) = x_n\downarrow$. Hence:

$$Me_{\mathrm{cpo}}(s) = \bigcup_{n\geqslant 0} x_n\downarrow .$$

We have to show that $\bigcup_{n\geqslant 0}\downarrow^n(x_n) = \bigcup_{n\geqslant 0} x_n\downarrow$.

$\subseteq$: is clear since $\downarrow^n(x_n)\subseteq x_n\downarrow$.

$\supseteq$: By the monotonicity of F_{cpo} we get

$$x_n\downarrow = g_n(s)\subseteq g_{n+1}(s) = x_{n-1}\downarrow .$$

Hence $x_0\sqsubseteq x_1\sqsubseteq x_2\sqsubseteq \ldots$ If $y\in x_n\downarrow$ then $y\sqsubseteq x_n$. Let

$$k = \max\{\rho(y), n\}.$$

Then $y\sqsubseteq x_k$ and $\rho(y)\leqslant k$. Hence $y\in\downarrow^k(x_k)$.

We conclude: $Me_{\mathrm{cpo}}(s) = \bigcup x_n\downarrow = \bigcup\downarrow^n(x_n) = \bar{\imath}(Me_{\mathrm{cms}}(s))$. $\square$

Note that in Theorem 4.5 we do not require the completedness of $Idl(D)$ as a metric space. Hence $\bar{\imath}$ is a mapping $\overline{D}\to\mathscr{P}_\downarrow(D)$. Nevertheless we have $\bar{\imath}(Me_{\mathrm{cms}}(s))\in Idl(D)$ for all $s\in\mathscr{L}$.

Remark 4.6. In the case that ρ is a finite weight the proof of Theorem 4.5 would be easier: By Lemma 4.2 it can be shown that

$$\bar{\imath}\circ F_{\mathrm{cms}}(f) = F_{\mathrm{cpo}}(\bar{\imath}\circ f)$$

for each function $F:\mathscr{L}\to\overline{D}$ and that F_{cpo} is contracting w.r.t. d_ρ^*. By Banach's fixed point theorem F_{cpo} has a unique fixed point. Since

$$\bar{\imath}\circ Me_{\mathrm{cms}} = \bar{\imath}\circ F_{\mathrm{cms}}(Me_{\mathrm{cms}}) = F_{\mathrm{cpo}}(i\circ Me_{\mathrm{cms}})$$

we get that $\bar{\imath}\circ Me_{\mathrm{cms}}$ is the unique fixed point of F_{cpo}. Hence $Me_{\mathrm{cpo}} = \bar{\imath}\circ Me_{\mathrm{cms}}$. $\square$

5. Related work

Various other authors have attempted to build a bridge between cpo and metrics. E.g. Matthews [13] introduces the notion of partial metrics and quasi metrics in order to obtain a topology that is not Hausdorff. He does not study completion and semantic operations. Smyth [16] introduces quasi-uniformities for the same propose.

We show the connection of the metric d_ρ^* on $Idl(D)$ and the metrics d_ϕ of [6]. In contrast to our approach, [6] only deals with countable posets. If $(D, \sqsubseteq)$ is a poset and $\phi : N_0 \to D$ a surjective mapping then the metric d_ϕ of [6] is given by:

$$d_\phi(I,J) = \frac{1}{1 + \min\{n : \phi(n) \in I\Delta J\}}$$

where

$$I\Delta J = I \setminus J \cup J \setminus I$$

Let us call two metrics d_1, d_2 on a set M to be equivalent iff the Cauchy sequences w.r.t. d_1 and d_2 are exactly the same. Our aim is to show the equivalence of d_ϕ and d_ρ^*. Then their completions are the same (more precisely: the underlying set is the same and the metrics induce the same topology). I.e. when the equivalence of d_ϕ and d_ρ^* is shown then the results of the earlier sections also hold for d_ϕ instead of d_ρ^*: We assume that D is endowed with the metric

$$d[\phi](x,y) = d_\phi(x\downarrow, y\downarrow).$$

Then $\imath : D \to Idl(D)$ is an isometric embedding of the metric space $(D, d[\phi])$ into the metric space $(Idl(D), d_\phi)$. Since d_ϕ and d_ρ^* are equivalent $d[\rho]$ and $d[\phi]$ are equivalent metrics on D. If $(\overline{D}, \bar{d})$ is the completion of $(D, d[\rho])$ then there exists an equivalent metric $\bar{d}_\phi$ on $\overline{D}$ such that $(\overline{D}, \bar{d}_\phi)$ is the completion of $(D, d[\phi])$. $(Idl(D), d_\rho^*)$ is complete if and only if $(Idl(D), d_\phi)$ is complete. In this case, the canonical extension $\bar{\imath} : \overline{D} \to Idl(D)$ of $\imath$ w.r.t. $d[\rho]$ and d_ρ^* coincides with the canonical extension of $\imath$ w.r.t. $d[\phi]$ and d_ϕ. By our results of Section 3, if $(Idl(D), d_\phi)$ is complete then $\bar{\imath} : (\overline{D}, \bar{d}_\phi) \to (Idl(D), d_\phi)$ is an isometric embedding.

If D is finite then also $Idl(D)$ is finite. Since the topology induced by a metric on a finite set is always the discrete metric we get in the finite case that all metrics on $Idl(D)$ are equivalent. Therefore in the following we only consider the case that D is infinite and countable.

Instead of the metric d_ϕ we consider the following metric d_ϕ^* which is equivalent to d_ϕ:

$$d_\rho^*(I,J) = \frac{1}{2^{\min\{n : \phi(n) \in I\Delta J\}}}$$

Remark 5.1. In [6] it is shown that whenever $\phi, \psi : \mathbb{N}_0 \to D$ are surjective mappings then the induced metrics d_ϕ and d_ψ are equivalent. Hence d_ρ^* and d_ψ^* are equivalent. Since D is infinite there exists a bijective mapping $\psi : \mathbb{N}_0 \to D$. Therefore we may assume that the underlying mapping $\phi : \mathbb{N}_0 \to D$ is bijective.

Lemma 5.2. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ and $\phi : \mathbb{N} \to D$ a surjective mapping. Then, each Cauchy sequence in $(Idl(D), d_\rho^*)$ is a Cauchy sequence in $(Idl(D), d_\phi^*)$.*

Proof. Let (I_n) be a Cauchy sequence in $(Idl(D), d_\rho^*)$. Let $\varepsilon > 0$ and $k_0 \geqslant 1$ such that

$$\frac{1}{2^{k_0}} < \varepsilon.$$

Let

$$m_0 = \max\{\rho(\phi(i)) : 0 \leqslant i \leqslant k_0\}.$$

Then for all $z \in D$ with $\rho(z) > m_0$: if $z = \phi(l)$ then $l \geqslant k_0$. Let $n_0 \geqslant 0$ such that

$$d_\rho^*(I_n, I_m) \leqslant \frac{1}{2^{m_0}} \quad \forall m, n \geqslant n_0.$$

Let $N = \max\{n_0, m_0\}$. Then for all $n, m \geqslant N$:

$$I_n[m_0] = I_m[m_0]$$

Let $z \in I_n \Delta I_m$ and $l \in \phi^{-1}(z)$. Then $\rho(z) > m_0$ and therefore $l \geqslant k_0$. We conclude:

$$\min\{l : \ \phi(l) \in I_n \Delta I_m\} \geqslant k_0$$

Therefore,

$$d_\rho^*(I_n, I_m) \leqslant \frac{1}{2^{k_0}} < \varepsilon.$$

I.e. (I_n) is a Cauchy sequence in $(Idl(D), d_\phi^*)$. $\square$

Lemma 5.3. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that for all $n \geqslant 0$ the set*

$$\{x \in D : \rho(x) \leqslant n\}$$

is finite. Let $\phi : \mathbb{N}_0 \to D$ be a surjective mapping. Then, each Cauchy sequence in $(Idl(D), d_\phi^*)$ is a Cauchy sequence in $(Idl(D), d_\rho^*)$.

Proof. By Remark 5.1 we may assume that ϕ is bijective. Let $\sigma : \mathbb{N} \to \mathbb{N}_0$ be given by

$$\sigma(n) = \min\{\rho(\phi(k)) : k > n\}.$$

Then $\sigma(0) \leqslant \sigma(1) \leqslant \cdots$. Let $\varepsilon > 0$ and $k_0 \geqslant 1$ with

$$\frac{1}{2^{k_0 - 1}} < \varepsilon.$$

Let $A = \{x \in D : \rho(x) < k_0\}$. By assumption, A is finite set. Since ϕ is bijective $\phi^{-1}(A)$ is finite. Let

$$r = \max \phi^{-1}(A).$$

Then for all $k > r$: $\phi(k) \notin A$, i.e. $\rho(\phi(k)) \geqslant k_0$. We conclude $\sigma(r) \geqslant k_0$. Let $N \geqslant 0$ such that

$$d^*_\phi(I_n, I_m) < \frac{1}{2^r} \quad \forall n, m \geq N.$$

Then for all $n, m \geqslant N$:

$$\min \{l: \ \phi(l) \in I_n \Delta I_m\} > r$$

Hence:

$$k_0 \leqslant \sigma(r) = \min \{\rho(\phi(l)) : l > r\}$$

We conclude that if $x \in I_n \Delta I_m$ and $l = \phi^{-1}(x)$ then $l > r$ and therefore $\rho(x) = \rho(\phi(l)) \geq k_0$. Hence

$$I_n[k_0 - 1] = I_m[k_0 - 1]$$

and therefore

$$d^*_\rho(I_n, I_m) \leqslant \frac{1}{2^{k_0 - 1}} < \varepsilon$$

for all $n, m \geqslant N$. I.e. (I_n) is a Cauchy sequence in $(Idl(D), d^*_\rho)$. $\square$

Remark 5.4. In Lemma 5.3 the condition that the sets $\{x \in D : \rho(x) \leqslant n\}$ are finite is necessary. Example: Consider the pointed poset $(D, \sqsubseteq)$ where

$$D = \{\bot, x_1, x_2, \ldots\} \quad \text{and} \quad \bot \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$$

Let $\phi : \mathbb{N}_0 \to D$ and $\rho : D \to \mathbb{N}_0$ be given by:

$$\phi(0) = \bot, \qquad \phi(n) = x_n, \qquad \rho(\bot) = 0, \qquad \rho(x_n) = 1.$$

Then ρ is finite length on $(D, \sqsubseteq)$ and d^*_ρ is the discrete metric on $Idl(D)$. If $m > n \geqslant 1$ then

$$x_n \downarrow \Delta\, x_m \downarrow = \{x_{n+1}, x_{n+2}, \ldots, x_m\}.$$

Hence

$$d^*_\rho(x_n \downarrow, x_m \downarrow) = \frac{1}{2^{n+1}}.$$

I.e. $(x_n \downarrow)_{n \geqslant 1}$ is a Cauchy sequence w.r.t. d^*_ϕ but not w.r.t. d^*_ρ. $\square$

By Lemmas 5.2 and 5.3 and the equivalence of d_ϕ and d_ϕ^* we get:

Theorem 5.5. *Let ρ be a finite length on a pointed poset $(D, \sqsubseteq)$ such that for all $n \geqslant 0$ the set*

$$\{x \in D : \rho(x) \leqslant n\}$$

is finite. Let $\phi : \mathbb{N}_0 \to D$ be a surjective mapping. Then d_ρ^* and d_ϕ are equivalent.

If $(D, \sqsubseteq)$ is a pointed poset and $\phi : \mathbb{N}_0 \to D$ a surjective mapping with

$$\phi(n) \sqsubseteq \phi(m) \Rightarrow n \leqslant m$$

then

$$\rho = \rho_\phi : D \to \mathbb{N}_0, \qquad \rho(x) = \min\{n \geqslant 0 : \phi(n) = x\}$$

is a finite length on $(D, \sqsubseteq)$ which satisfies the conditions of Theorem 5.5. Hence the metric d_ϕ on $Idl(D)$ in the sense of [6] is equivalent to the metric d_ρ^*. I.e. in this case the approach of [6] is a special case of our approach.

References

[1] C. Baier and M.E. Majster-Cederbaum, Denotational semantics in the cpo and metric approach, *Theoret. Comput. Sci.* **135** (1994) 171–220.

[2] C. Baier and M.E. Majster-Cederebaum, Construction of a cms on a given cpo, accepted for publication in Acta Informatica.

[3] C. Baier and M.E. Majster-Cederbaum, The connection between initial and unique solutions of domain equations in the partial order and metric approach, submitted for publication.

[4] J.W. de Bakker and J.J. Ch. Meyer, Metric semantics for concurrency, *BIT* **28** (1988) 504–529.

[5] J.W. de Bakker and J.I. Zucker, Processes and the denotational semantics of concurrency, *Inform. and Control* **54** (1/2) (1982) 70–120.

[6] G. Comyn and M. Dauchet, Metric approximations in ordered domains in: M. Nivat and J. Reynolds, eds.: *Algebraic Methods in Semantics* (Cambridge University Press, Cambridge 1985).

[7] G. Gierz, H. Hofmann, K. Keimel, J. Lawson, M. Mislove and D. Scott; *A Compendium of Continuous Lattices* (Springer, Berlin, 1980).

[8] U. Goltz and R. Loogen, Modelling nondeterministic concurrent processes with event structures, *Fundam. Inform.* **14** (1) (1991) 39–73.

[9] C.A. Gunter, *Semantics of Programming Languages* (MIT Press, Cambridge, MA, 1992).

[10] K. Kuratowski, *Topology* (Academic Press, New York, 1966).

[11] M.Z. Kwiatkowska, On these constructions of infinite traces, Techn. Report No. CSD-48, University of Leicester (1991).

[12] M. Majster-Cederbaum and F. Zetzsche, The comparison of a cpo-based with a cms-based semantics for CSP, *Theoret. Comput. Sci.* **124** (1994) 1–40.

[13] S. Matthews, The Cycle Contracting Mapping Theorem, Research Report 228, Department of Computer Science, University of Warwick (1992).

[14] A. Mazurkiewicz, Basic notions of trace theory, in: *Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency*, Lecture Notes in Computer Science, Vol. 354 (Springer, Berlin, 1989).

[15] R. Milner, *Communication and Concurrency* (Prentice-Hall, Englewood Cliffs, NJ, 1989).

[16] M.B. Smyth, Quasi-uniformities: reconciling domains with metric spaces, in: *Proc. Third Workshop on the Mathematical Foundations of Programming Language Semantics*, Lecture Notes in Computer Science, Vol. 298 (Springer, Berlin, 1988) 236–253.

[17] G. Winskel, Event structure semantics for *CCS* and related languages, Proc. ICALP 82, Lecture Notes in Computer Science, Vol. 140 (Springer, Berlin, 1982) 561–576.

[18] G. Winskel, Synchronisation trees, *Theoret. Comput. Sci.* **34** (1984) 33–82.